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Orthogonal Polynomials and Christoffel Functions for $\operatorname{Exp}(-|X|^{\alpha})$, $\alpha \leqslant 1$

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Let $W_\alpha(x) := \exp(-|x|^\alpha)$, $x \in \mathbb{R}$, $\alpha > 0$. For $\alpha \leqslant 1$, we obtain upper and lower bounds for the Christoffel functions for the weight W_α^2 over the whole Mhaskar–Rahmanov–Saff interval, and deduce inequalities for spacing of zeros of orthogonal polynomials for W_α^2. Then we deduce bounds for orthogonal polynomials for the weight W_α^2. These results complement recent results of the authors treating a large class of weights including W_α^2, $\alpha > 1$. © 1995 Academic Press, Inc.

1. Introduction and Results

Let $W^2 := e^{-2Q}$, where $Q: \mathbb{R} \to \mathbb{R}$ is even, continuous, and of "smooth polynomial growth" at infinity. Such a weight is often called a *Freud weight* [19], and perhaps the archetypal example is

$$W_\alpha(x) := \exp(-|x|^\alpha), \qquad \alpha > 0. \tag{1.1}$$

Corresponding to the weight W^2, we can define *orthonormal polynomials*

$$p_n(x) := p_n(W^2, x) = \gamma_n x^n + \cdots, \qquad \gamma_n > 0, n \geqslant 0,$$

satisfying

$$\int_{-\infty}^{\infty} p_n(W^2, x)\, p_m(W^2, x)\, W^2(x)\, dx = \delta_{mn}, \qquad m, n \geqslant 0.$$

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Recently, the authors [10] established bounds for $p_n(W^2, x)$ for a class of Freud weights that includes W_α^2, $\alpha > 1$. The purpose of this paper is to establish complementary results for the case $\alpha \leqslant 1$. Our methods are similar to those in [10], but additional technical difficulties arise. Consequently, we have decided to restrict ourselves to the weights W_α^2, though the methods can treat more general Freud weights.

Here, as in [10], estimates for the *Christoffel function* play a crucial role. Recall that if $\mathscr{P}_n$ denotes the class of polynomials of degree $\leqslant n$, then

$$\lambda_n(W^2, x) := \inf_{P \in \mathscr{P}_{n-1}} \int_{-\infty}^{\infty} (PW)^2(t)\, dt/P^2(x) \tag{1.2}$$

$$= 1 \Big/ \sum_{j=0}^{n-1} p_j^2(W^2, x). \tag{1.3}$$

See [19] for a survey of the importance of Christoffel functions.

To state our results, we need the *Mhaskar–Rahmanov–Saff* number a_u [16, 17], the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 a_u t Q'(a_u t)\, dt/\sqrt{1-t^2}, \qquad u > 0. \tag{1.4}$$

For the weight $W_\alpha(x)$, we have $Q(x) = |x|^\alpha$, and

$$a_n(W_\alpha) = (n/\lambda_\alpha)^{1/\alpha}, \qquad n \geqslant 1, \tag{1.5}$$

where [16]

$$\lambda_\alpha = \Gamma(\alpha)/[2^{\alpha-2}\Gamma(\alpha/2)^2]. \tag{1.6}$$

Throughout $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, and $P \in \mathscr{P}_n$. We use $\sim$ in the following sense: If $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are sequences of non-zero real numbers, we write

$$b_n \sim c_n,$$

if there exist $C_1, C_2 > 0$ such that

$$C_1 \leqslant b_n/c_n \leqslant C_2, \qquad n \geqslant 1.$$

Similar notation is used for functions and sequences of functions.

Given $0 < \alpha \leqslant 1$, and $n \geqslant 1$, we define a function $\Lambda_n(x) := \Lambda_n(\alpha, x)$ as follows: For $|x| \leqslant a_n/2$, set

$$\Lambda_n(x) := \begin{cases} (1+|x|)^{1-\alpha}, & \alpha < 1 \\ 1/\log[\pi n/(1+|x|)], & \alpha = 1 \end{cases} \tag{1.7}$$

and for $|x| \geqslant a_n/2$,

$$\Lambda_n(x) := n^{1/\alpha - 1} \max\{n^{-2/3}, 1 - |x|/a_n\}^{-1/2}. \tag{1.8}$$

We remark that the breakpoint $a_n/2$ is just for definiteness: We could have used σa_n for any $0 < \sigma < 1$, as our breakpoint, since the ratio of the right-hand sides of (1.7), (1.8) ~ 1 in $[\delta a_n, \varepsilon a_n]$ for any fixed $0 < \delta < \varepsilon < 1$.

Following is our result for Christoffel functions:

THEOREM 1.1. *Let $0 < \alpha \leqslant 1$ and $L > 0$. Then uniformly for $n \geqslant 1$ and $|x| \leqslant a_n(1 + Ln^{-2/3})$, we have*

$$\lambda_n(W^2, x) \sim \Lambda_n(x)\, W_\alpha^2(x). \tag{1.9}$$

Moreover, there exists $C > 0$ such that for $n \geqslant 1$ and all $x \in \mathbb{R}$,

$$\lambda_n(W_\alpha^2, x) \geqslant C\Lambda_n(x)\, W_\alpha^2(x). \tag{1.10}$$

Remarks. (a) The lower bound (1.10) was proved in [10]. We use the method of [10] to prove the upper bound implicit in (1.9) for $|x| \in [\varepsilon a_n, a_n(1 + Ln^{-2/3})]$, any $0 < \varepsilon < 1$, but that method breaks down for $|x| \leqslant \varepsilon a_n$. To prove the upper bounds for $|x| \leqslant \varepsilon a_n$, we use the method that Freud, Giroux, and Rahman employed for $\alpha = 1$ in [7]: They established (1.9) for $\alpha = 1$ and the range $|x| \leqslant \varepsilon a_n$, some $\varepsilon > 0$.

(b) It is a well known consequence [2, 5, 20] of the indeterminacy of the moment problem for $\alpha < 1$ that $\lambda_n(W_\alpha^2, x)$ does not decay to 0 as $n \to \infty$, or equivalently

$$K_\alpha(x) := \sum_{j=0}^{\infty} p_j^2(W_\alpha^2, x) < \infty.$$

In fact, Theorem 1.1 implies that

$$K_\alpha(x)\, W_\alpha^2(x) \sim (1 + |x|)^{\alpha - 1}, \qquad \text{uniformly for } x \in \mathbb{R}. \tag{1.11}$$

The order and type of the entire function $K_\alpha(x)$ have been investigated by various authors; see, for example, [2, 3].

We can deduce results on the zeros of the orthonormal polynomial $p_n(W_\alpha^2, x)$, which we order as

$$-\infty < x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} < \infty.$$

COROLLARY 1.2. *Let $0<\alpha\leqslant 1$. Then there exists C_1 such that*

(a) *For $n\geqslant 1$,*

$$|x_{1n}/a_n-1|\leqslant C_1 n^{-2/3}. \tag{1.12}$$

(b) *Uniformly for $n\geqslant 2$ and $2\leqslant j\leqslant n-1$,*

$$x_{j-1,n}-x_{j+1,n}\sim A_n(x_{jn}). \tag{1.13}$$

Remarks. (a) For α a positive even integer, sharper asymptotics are known for x_{1n} [14].

(b) We can probably deduce a similar result for $x_{jn}-x_{j+1,n}$ with additional work; see [4].

COROLLARY 1.3. *Let $0<\alpha\leqslant 1$ and $\varepsilon\in(0,1)$. Then uniformly for $n\geqslant 1$ and j such that $|x_{jn}|\geqslant\varepsilon a_n$,*

$$\begin{aligned} n^{1/\alpha-1}&|p_n'(W_\alpha^2,x_{jn})|\,W_\alpha(x_{jn})\\ &\sim|p_{n-1}(W^2,x_{jn})|\,W(x_{jn})\\ &\sim n^{-1/(2\alpha)}\max\{n^{-2/3},1-|x_{jn}|/a_n\}^{1/4}. \end{aligned} \tag{1.14}$$

The reason for our restriction $|x|\geqslant\varepsilon\alpha_n$ is that we cannot obtain correct upper bounds for a certain function $A_n(x)$ for $|x|\leqslant\varepsilon a_n$; see Section 6.

COROLLARY 1.4. *Let $0<\alpha\leqslant 1$ and $\varepsilon\in(0,1)$. Then for $n\geqslant 1$ and $|x|\in[\varepsilon a_n,a_n]$,*

$$|p_n(W_\alpha^2,x)|\,W_\alpha(x)\leqslant Cn^{-1/(2\alpha)}\max\{n^{-2/3},1-|x|/a_n\}^{-1/4}. \tag{1.15}$$

Remarks. (a) Again, the restrictions on the range of x in (1.15) arise from our inability to investigate the behaviour of a certain function. Using the asymptotics in [11, pp. 187, 209] for weights that are the reciprocals of an entire function, and Korous type identities, we can obtain "correct" upper bounds for $p_n(W_\alpha^2,x)$ for $|x|\geqslant a_n n^{-1/3+\delta}$, any $\delta>0$. However, this involves substantial effort, and does not provide bounds for the complete range, so is omitted.

(b) E. A. Rahmanov [22] has informed the authors that he believes asymptotics can be proved for $p_n(W_\alpha^2,x)$ in $[-\sigma a_n,\sigma a_n]$, any fixed $\sigma\in(0,1)$. Such asymptotics will imply

$$\|p_n(W_\alpha^2,\cdot)\,W_\alpha\|_{L_\infty[-\sigma a_n,\sigma a_n]}\leqslant Ca_n^{-1/2},\qquad n\geqslant 1.$$

Together with Corollary 1.4, the methods in Section 6 or in [10] will give

$$\| p_n(W_\alpha^2, x)\, W_\alpha(x) |1-|x|/a_n|^{1/4} \|_{L_\infty(\mathbb{R})} \sim a_n^{-1/2}, \qquad n \geq 1,$$

and

$$\| p_n(W_\alpha^2, \cdot)\, W_\alpha(\cdot) \|_{L_\infty(\mathbb{R})} \sim a_n^{-1/2} n^{1/6}, \qquad n \geq 1.$$

At least for $\alpha = 1$, we can prove this:

COROLLARY 1.5.

$$\| p_n(W_1^2, \cdot)\, W_1 \|_{L_\infty(\mathbb{R})} \sim n^{-1/2+1/6}, \qquad n \geq 1. \tag{1.16}$$

This paper is organised as follows: In Section 2, we discretise a potential, and hence estimate a certain L_∞ Christoffel function. In Section 3, we obtain upper bounds for $\lambda_n(W_\alpha^2, x)$ for the range $|x| \in [\varepsilon a_n, a_n(1+Ln^{-2/3})]$ and in Section 4, we obtain upper bounds for the range $|x| \leq \varepsilon a_n$, thereby completing the proof of Theorem 1.1. In Section 5, we prove Corollary 1.2 on the zeros of $p_n(W_\alpha^2, x)$, and in Section 6, we prove Corollaries 1.3–1.5.

2. THE SUP-NORM CHRISTOFFEL FUNCTIONS

Given a weight $W: \mathbb{R} \to \mathbb{R}$, we let

$$\lambda_{n,\infty}(W, x) := \inf_{P \in \mathscr{P}_{n-1}} \|PW\|_{L_\infty(\mathbb{R})}/|P|(x), \qquad n \geq 1, x \in \mathbb{R}, \tag{2.1}$$

denote the *sup-norm Christoffel function* for W. In this section, we obtain the following result, which will be applied in the next section to derive upper bounds for ordinary Christoffel functions:

THEOREM 2.1. *Let $\alpha > 0$ and $L > 0$. For $n \geq 1$, set*

$$\mathscr{I}_n := \{x : |x| \leq a_n(1+Ln^{-2/3})\}, \tag{2.2}$$

where $a_n = a_n(W_\alpha)$ is defined by (1.5) *and* (1.6). *Then*

$$\lambda_{n,\infty}(W_\alpha, x) \sim W_\alpha(x), \tag{2.3}$$

uniformly for $x \in \mathscr{I}_n$ and $n \geq 1$.

Note that for $\alpha > 1$, Theorem 2.1 is a special case of Theorem 1.6 in [10], so we concentrate on the case $\alpha \leq 1$. Obviously $\lambda_{n,\infty}(W_\alpha, x) \geq$

$W_\alpha(x)$. Thus it suffices to construct, for any $x_0 \in \mathscr{J}_n$, a polynomial $S_n = S_{n, x_0} \in \mathscr{P}_n$, such that

$$\|S_n W_\alpha\|_{L_\infty(\mathbb{R})} \leqslant C_1, \tag{2.4}$$

while

$$|(S_n W_\alpha)(x_0)| \geqslant C_2, \tag{2.5}$$

where C_1 and C_2 depend on α, L, but not on n or $x_0 \in J_n$ (We may use $S_n \in \mathscr{P}_n$ instead of $S_n \in \mathscr{P}_{n-1}$, since $a_{n-1}/a_n = 1 + O(n^{-1})$, by (1.5).)

First let us reformulate our task. We need some potential theory related to W_α:

LEMMA 2.2. *Let* $\alpha > 0$.

(a) *Define for* $x \in [-1, 1] \backslash \{0\}$,

$$\mu(x) := \frac{2}{\pi^2} \alpha \lambda_\alpha^{-1} \int_0^1 \frac{\sqrt{1-x^2}}{\sqrt{1-s^2}} \frac{s^\alpha - |x|^\alpha}{s^2 - x^2} \, ds. \tag{2.6}$$

Then

$$\mu(x) > 0 \quad \text{in} \quad [-1, 1] \backslash \{0\} \quad \text{and} \quad \int_{-1}^1 \mu(t) \, dt = 1. \tag{2.7}$$

(b) *Define for* $z \in \mathbb{C}$,

$$U(z) := \int_{-1}^1 \log |z - t| \, \mu(t) \, dt - \lambda_\alpha^{-1} |z|^\alpha + \chi_\alpha, \tag{2.8}$$

where

$$\chi_\alpha := \frac{2}{\pi} \lambda_\alpha^{-1} \int_0^1 \frac{t^\alpha}{\sqrt{1-t^2}} \, dt + \log 2.$$

Then for $x \in [-1, 1]$,

$$U(x) = 0;$$

and

$$\exp\left(-n \int_{-1}^1 \log |x - t| \, \mu(t) \, dt\right) = W_\alpha(a_n x) \exp(n \chi_\alpha). \tag{2.10}$$

Furthermore,

$$U(x) \leqslant 0, \qquad x \in \mathbb{R}, \tag{2.11}$$

and

$$|nU(x)| \leqslant C, \qquad x \in \mathscr{J}_n, \tag{2.12}$$

where $\mathscr{J}_n$ *is defined by* (2.2) *and* $C = C(L)$.

Proof. These statements are well known and appear (in various forms) in [12, 16, 21]. For our purposes, a convenient reference is Lemma 7.1 in [10], applied in the special case $Q(x) := |x|^\alpha$ and $R = a_n = (n/\lambda_\alpha)^{1/\alpha}$. (Note that there is a missing minus sign in the exponential term in (7.11) in [10]). ∎

Assume now that for any $x_0 \in \mathbb{R}$, there exists a polynomial $P_n = P_{n, x_0} \in \mathscr{P}_n$ such that

$$|P_n(x)| \leqslant C_1 \exp\left\{n \int_{-1}^{1} \log|x - t| \, \mu(t) \, dt\right\}, \qquad x \in \mathbb{R}, \tag{2.13}$$

and

$$|P_n(x_0/a_n)| \geqslant C_2 \exp\left\{n \int_{-1}^{1} \log|x_0/a_n - t| \, \mu(t) \, dt\right\}, \tag{2.14}$$

where C_1 and C_2 are constants independent of n and x_0. Then, on setting

$$S_n(x) := P_n(x/a_n) \exp(n\chi_\alpha),$$

we deduce (by (2.8) to (2.12)) that these S_n satisfy (2.4) and (2.5).

Therefore, in order to prove Theorem 2.1, it remains to construct P_n as above. Such a construction was carried out in our paper [10, Theorem 9.1], for a large class of weights that includes W_α, $\alpha > 1$. For $\alpha \leqslant 1$, the same method applies, but the details become more cumbersome. Here we use another method that is due to V. Totik [13, 24] as it simplifies the estimation.

THEOREM 2.3. *Let $d\sigma$ be a positive Borel measure on $[a, b] \subset \mathbb{R}$ that satisfies*

$$\int d\sigma = 1, \tag{2.15}$$

and let

$$U^\sigma(z) := \int \log|z - t| \, d\sigma(t) \tag{2.16}$$

be the corresponding potential. Define $a = t_0 < t_1 < \cdots < t_n = b$ *by*

$$\int_{I_j} d_\sigma = \frac{1}{n}, \qquad I_j := [t_j, t_{j+1}], 0 \leqslant j \leqslant n-1. \tag{2.17}$$

Assume that the following conditions hold:

(a) *Uniformly for* $0 \leqslant j \leqslant n-1$,

$$|I_j| \sim |I_{j+1}|, \tag{2.18}$$

where $|I_j| := t_{j+1} - t_j$.

(b) *There exists* $C_1 > 0$ *such that uniformly for* $0 \leqslant j \leqslant n-1$, $x \in I_j$,

$$n \int_{I_j} \log\left(\frac{|x-t|}{|I_j|}\right) d\sigma(t) \geqslant -C_1. \tag{2.19}$$

(c) *There exists* $C_2 > 0$ *such that uniformly for* $0 \leqslant k \leqslant n-1$,

$$\sum_{j \leqslant k-2} \frac{|I_j|^2}{|t_{j+1} - t_k|^2} + \sum_{j \geqslant k+2} \frac{|I_j|^2}{|t_j - t_{k+1}|^2} \leqslant C_2. \tag{2.20}$$

Then, given any $x_0 \in \mathbb{R}$, *one can find a polynomial* $P_n = P_{n, x_0} \in \mathscr{P}_n$ *that satisfies*

$$|P_n(x)| \leqslant C_3 \exp(nU^\sigma(x)), \qquad x \in \mathbb{R}, \tag{2.21}$$

and

$$|P_n(x_0)| \geqslant \tfrac{1}{3} \exp(nU^\sigma(x_0)). \tag{2.22}$$

The constant C_3 *in* (2.21) *depends only on the constants* C_1, C_2 *in* (2.19), (2.20) *and on the constants implicit in the* $\sim$ *relation* (2.18).

Proof. Given $x_0 \in \mathbb{R}$, we construct P_n as follows:

Case I. $x_0 \notin [a, b]$ or $x_0 = t_j$ for some $0 \leqslant j < n$. Then define $\xi_j \in I_j$ by

$$\int_{I_j} (t - \xi_j)\, d\sigma(t) = 0, \qquad 0 \leqslant j \leqslant n-1. \tag{2.23}$$

Case II. $t_{j_0} < x_0 < t_{j_0+1}$ for some $0 \leqslant j_0 \leqslant n-1$. Then define ξ_j by (2.23), if $j \neq j_0$. As to ξ_{j_0}, this can be chosen arbitrarily in I_{j_0}, subject to the restriction

$$|x_0 - \xi_{j_0}| \geqslant \tfrac{1}{3} |I_{j_0}|. \tag{2.24}$$

With the above choice of ξ_j's, define

$$P_n(x) := \prod_{j=0}^{n-1} (x-\xi_j). \tag{2.25}$$

We claim that this P_n satisfies (2.21) and (2.22). The proof of (2.22) is particularly simple. We have by (2.17) and (2.24),

$$\begin{aligned} nU^\sigma(x) - \log|P_n(x)| &= \sum_{j=0}^{n-1} n \int_{I_j} \log\left|\frac{x-t}{x-\xi_j}\right| d\sigma(t) \\ &=: \sum_{j=0}^{n-1} L_j(x). \end{aligned} \tag{2.26}$$

The function $\log|x_0 - t|$ is concave as a function of t on I_j, provided $x_0 \notin (t_j, t_{j+1})$. Thus

$$\log|x_0 - t| \leqslant \log|x_0 - \xi_j| + \frac{t-\xi_j}{\xi_j - x_0}, \qquad t \in I_j.$$

Therefore the choice (2.23) of ξ_j ensures that $L_j(x_0) \leqslant 0$. If $x_0 \in (t_{j_0}, t_{j_0+1})$, we have by (2.24),

$$\left|\frac{x_0 - t}{x_0 - \xi_j}\right| \leqslant 3, \qquad t \in I_{j_0},$$

so that $L_{j_0}(x) \leqslant \log 3$. This proves (2.22).

To prove (2.21), we need to show (see (2.26)) that

$$\sum_{j=0}^{n-1} L_j(x) \geqslant -C, \qquad x \in \mathbb{R}. \tag{2.27}$$

Since the left-hand side of (2.26) represents a function that is harmonic in $\bar{\mathbb{C}} \setminus [a, b]$, it suffices to establish (2.27) for $x \in [a, b]$. So, let $x \in I_{j^*}$ for some $0 \leqslant j^* \leqslant n-1$. First, note that the condition (2.19) implies that every single term in (2.27) is bounded below by $-C_1$. In particular, we have

$$L_j(x) \geqslant -C_1, j = j^*; \qquad j^* \pm 1; j_0. \tag{2.28}$$

Next, for $j \neq j^*, j^* \pm 1$, we obtain from condition (2.18) that

$$\frac{\xi_j - t}{x - \xi_j} \geqslant -\beta > -1, \qquad t \in I_j,$$

with some $0<\beta<1$, independent of $x \in I_{j^*}$ and of j, n. Therefore, we may write for $t \in I_j$,

$$\log\left|\frac{x-t}{x-\xi_j}\right| = \log\left|1+\frac{\xi_j-t}{x-\xi_j}\right| \geqslant \frac{\xi_j-t}{x-\xi_j} - \frac{1}{2(1-\beta)^2}\left(\frac{\xi_j-t}{x-\xi_j}\right)^2.$$

If j is also different from j_0, we obtain by (2.23) that

$$\begin{aligned} L_j(x) &\geqslant -\frac{n}{2(1-\beta)^2}\int_{I_j}\left(\frac{\xi_j-t}{x-\xi_j}\right)^2 d\sigma(t) \\ &\geqslant -\frac{n}{2(1-\beta)^2}\int_{I_j}\left(\frac{|I_j|}{x-\xi_j}\right)^2 d\sigma(t) \\ &= -\frac{1}{2(1-\beta)^2}\left(\frac{|I_j|}{x-\xi_j}\right)^2, \end{aligned}$$

provided $j \neq j^*; j^* \pm 1; j_0$. Thus (recall (2.28)), in order to prove (2.27), it remains to show that

$$\sum_{j \neq j^*; j^* \pm 1; j_0} \left(\frac{|I_j|}{x-\xi_j}\right)^2 \leqslant C.$$

Since $x \in I_{j^*}$, $\xi_j \in I_j$, this inequality follows from condition (2.20). ∎

Theorem 2.3 gives us the desired relations (2.13), (2.14) provided we can show that the measure $d\sigma(t)=\mu(t)\,dt$ satisfies conditions (a)–(c) of Theorem 2.3. From (2.6), we easily deduce that

(i) For $\alpha>1$,

$$\mu(t) \sim \sqrt{1-t^2}, \qquad |t|<1.$$

(ii) For $\alpha=1$,

$$\mu(t) \sim \begin{cases} \sqrt{1-t^2}, & \frac{1}{4} \leqslant |t| < 1. \\ \log 1/|t|, & 0<|t| \leqslant \frac{1}{2}. \end{cases}$$

(iii) For $0<\alpha<1$,

$$\mu(t) \sim \sqrt{1-t^2}, \qquad \tfrac{1}{4} \leqslant |t| < 1, \tag{2.29}$$

$$\mu(t) \sim |t|^{\alpha-1}, \qquad 0<|t|<\tfrac{1}{2}. \tag{2.30}$$

We shall only consider the case (iii) and provide the reader a few details. Note that μ is an even function. Assume, for definiteness, that n is even. Then $t_{n/2}=0$ and (2.30) yields

$$\frac{j-n/2}{n} = \int_0^{t_j} \mu \sim t_j^{\alpha}$$

provided

$$j \in J_1 := \{j : j > n/2 \text{ and } 0 < t_j \leq \tfrac{1}{2}\}.$$

Therefore, uniformly for $j, j+1 \in J_1$, we have

$$t_{j+1}/t_j \sim 1. \tag{2.31}$$

This relation combined with

$$C_1 \, |I_j| \, t_{j+1}^{\alpha-1} \leq \int_{I_j} \mu = \frac{1}{n} \leq C_2 \, |I_j| \, t_j^{\alpha-1}$$

yields

$$n t_j^{\alpha-1} \, |I_j| \sim 1, j \in J_1, \tag{2.32}$$

and therefore (by (2.31)),

$$|I_j| \sim |I_{j+1}| \qquad \text{for} \quad j, j+1 \in J_1.$$

Also,

$$|I_{n/2}| \sim |I_{n/2+1}| \sim n^{-1/\alpha}. \tag{2.33}$$

Thus the condition (2.18) holds for $j \in J_1 \cup \{n/2\}$. Similar reasoning (based on (2.29)) shows that uniformly for j, $j+1 \in J_2 := \{j : j > n/2, \ 1 > t_j \geq \frac{1}{4}\}$, we have

$$\frac{1 - t_{j+1}^2}{1 - t_j^2} \sim 1; \qquad n \sqrt{1 - t_j^2} \, |I_j| \sim 1. \tag{2.34}$$

Also,

$$|I_{n-1}| \sim |I_{n-2}| \sim n^{-2/3}. \tag{2.35}$$

Thus, (2.18) holds for $j \in J_2$. Since J_1, J_2 overlap and since μ is even, we have verified (2.18). Next, we turn to (2.19). For $j = n/2$, $x \in I_{n/2}$, we have

$$\begin{aligned}
n \int_{I_{n/2}} &= n \int_0^{|I_{n/2}|} \log \left| \frac{x-t}{|I_{n/2}|} \right| \mu(t) \, dt \\
&\geq Cn \int_0^{|I_{n/2}|} \log \left| \frac{x-t}{|I_{n/2}|} \right| t^{\alpha-1} \, dt \qquad \text{(since the integrand is negative)} \\
&= Cn \, |I_{n/2}|^{\alpha} \int_0^1 \log |y-s| \, s^{\alpha-1} \, ds,
\end{aligned}$$

by the substitution $x=y\,|I_{n/2}|$ $(0<y<1)$, $t=s\,|I_{n/2}|$. The last integral is uniformly bounded for $0\leqslant y\leqslant 1$. Taking into account (2.33), we have shown that (2.19) holds for $j=n/2$. For $j\in J_1$, we obtain

$$n\int_{I_j}\log\left|\frac{x-t}{|I_j|}\right|\mu(t)\,dt\geqslant Cnt_j^{\alpha-1}\int_{I_j}\log\left|\frac{x-t}{|I_j|}\right|dt$$

$$\geqslant Cnt_j^{\alpha-1}\,|I_j|\int_{-1}^{1}\log|s|\,ds,$$

by the substitution $x-t=s\,|I_j|$. Applying (2.32), we obtain (2.19) uniformly for $j\in J_1$. The case $j\in J_2$ is treated similarly, by using (2.34) and (2.35).

Finally, we prove (2.20). This is equivalent (in view of (2.31), (2.32), (2.34), (2.35)) to

$$\frac{1}{n}\int_{|t-t_k|\geqslant|I_k|}\frac{dt}{(t-t_k)^2\,\mu(t)}\leqslant C_2.$$

Assume, for example, that $k=n/2$. Then the last integral is bounded from above by

$$\frac{1}{n}\int_{Cn^{-1/\alpha}}^{1/2}\frac{dt}{t^2\cdot t^{\alpha-1}}+\frac{1}{n}\int_{1/2}^{1}\frac{dt}{t^2\sqrt{1-t^2}}=O(1).$$

Similarly, using (2.32)–(2.35), we obtain (2.20) uniformly in $0\leqslant k\leqslant n-1$. This completes the proof.

3. Upper Bounds for Christoffel Functions, I

In this section, we obtain upper bounds for the Christoffel function $\lambda_n(W_\alpha^2, x)$, for $|x|\in[\varepsilon a_n, a_n(1+Ln^{-2/3})]$, for any fixed $\varepsilon\in(0,1)$ and then deduce Theorem 1.1(a) for this range. The proof follows very closely those in [10], but we present the details for the reader's convenience. First, we recall an infinite–finite range inequality from [10]:

Lemma 3.1. *Let $0<p\leqslant\infty$ and $K>0$. Then there exist N, $C>0$ such that for $n\geqslant N$ and $P\in\mathcal{P}_n$,*

$$\|PW_\alpha\|_{L_p(\mathbb{R})}\leqslant C\,\|PW_\alpha\|_{L_p(|x|\leqslant a_n(1-Kn^{-2/3}))}. \tag{3.1}$$

Proof. This is a special case of Theorem 1.8 in [10]. ∎

Proof of Theorem 1.1(a) *for the range* $|x| \in [\varepsilon a_n, a_n(1+Ln^{-2/3})]$. First, note that by Lemma 3.1, there exists $C>0$ such that

$$\|PW_\alpha\|_{L_2(\mathbb{R})} \leqslant C\, \|PW_\alpha\|_{L_2[-a_n, a_n]}, \qquad P \in \mathscr{P}_n,\ n \geqslant 1.$$

Hence for any $m \leqslant n-1$,

$$\begin{aligned}
\lambda_n(W_\alpha^2, x)/W_\alpha^2(x) &\leqslant C^2 \inf_{P \in \mathscr{P}_{n-1}} \int_{-a_n}^{a_n} (PW_\alpha)^2(t)\, dt/(PW_\alpha)^2(x) \\
&\leqslant C^2 \,[\inf_{P \in \mathscr{P}_{m-1}} \|PW_\alpha\|_{L_\infty(\mathbb{R})}/|PW_\alpha|(x)]^2 \\
&\quad \times \inf_{P \in \mathscr{P}_{n-m}} \int_{-a_n}^{a_n} P^2(t)\, dt/P^2(x) \\
&= C^2\, [\lambda_{m,\infty}(W_\alpha, x)/W_\alpha, x)]^2\, a_n \lambda_{n-m+1}(u, x/a_n),
\end{aligned}$$

where $u \equiv 1$ is the Legendre weight on $[-1, 1]$. By standard estimates for the Christoffel function of the Legendre weight [18, pp. 107–108],

$$\lambda_l(u, x) \leqslant \frac{C}{l} \max\{1-|x|, l^{+2}\}^{+1/2}, \qquad x \in [-1, 1].$$

Since

$$\lambda_l(u, x) = 1 \Big/ \sum_{j=0}^{l-1} p_j^2(u, x)$$

is a decreasing function of $x \in (1, \infty)$, the above estimate also holds outside $[-1, 1]$. Hence, we obtain for $x \in \mathbb{R}$,

$$\begin{aligned}
\lambda_n(W_\alpha^2, x)/W_\alpha^2(x) &\leqslant C_1\, [\lambda_{m,\infty}(W_\alpha, x)/W_\alpha(x)]^2 \\
&\quad \times \frac{a_n}{n-m} \max\left\{1 - \frac{|x|}{a_n}, \frac{1}{(n-m)^2}\right\}^{1/2}.
\end{aligned} \tag{3.2}$$

We distinguish two ranges of x:

(I) $\varepsilon a_n \leqslant |x| \leqslant a_{n(1-2n^{-2/3})}$. In this case, we choose an integer m such that

$$a_{m-1} < |x| \leqslant a_m.$$

Then for n large enough,

$$Cn \leqslant m \leqslant n(1-n^{-2/3}), \tag{3.3}$$

where C of course depends on ε. Recalling that a_n is given by (1.5), we see that

$$1-\frac{|x|}{a_n}\sim 1-\frac{a_m}{a_n}\sim 1-\frac{m}{n}.$$

In particular, together with (3.3), this implies that

$$\frac{1}{(n-m)^2}\leqslant n^{-2/3}\leqslant 1-\frac{m}{n}\sim 1-\frac{|x|}{a_n},$$

and hence,

$$\frac{a_n}{n-m}\max\left\{1-\frac{|x|}{a_n},\frac{1}{(n-m)^2}\right\}^{1/2}$$
$$\sim\frac{a_n}{n}\frac{1}{1-m/n}\left(1-\frac{|x|}{a_n}\right)^{1/2}\sim\frac{a_n}{n}\left(1-\frac{|x|}{a_n}\right)^{-1/2}.$$

This $\sim$ relation, (3.2), and Theorem 2.1 show that for $\varepsilon a_n\leqslant|x|\leqslant a_{n(1-2n^{-2/3})}$,

$$\lambda_n(W_\alpha^2,x)/W_\alpha^2(x)\leqslant C_2\frac{a_n}{n}\left(1-\frac{|x|}{a_n}\right)^{-1/2}. \tag{3.4}$$

(II) $a_{n(1-2n^{-2/3})}\leqslant|x|\leqslant a_n(1+Ln^{-2/3})$. In this case, we choose

$$m:=n-\langle n^{1/3}\rangle,$$

where $\langle x\rangle$ denotes the greatest integer $\leqslant x$. Then, we see that

$$1-\frac{|x|}{a_n}\leqslant C_1n^{-2/3}\sim\frac{1}{(n-m)^2},$$

so that

$$\frac{a_n}{n-m}\max\left\{1-\frac{|x|}{a_n},\frac{1}{(n-m)^2}\right\}^{1/2}\leqslant C_3\frac{a_n}{(n-m)^2}\leqslant C_4a_nn^{-2/3}. \tag{3.5}$$

Finally,

$$|x|/a_m\leqslant(a_n/a_m)(1+Ln^{-2/3})\leqslant 1+Kn^{-2/3},$$

some $K>0$. Then (3.2), (3.5), and Theorem 2.1 show that

$$\lambda_n(W_\alpha^2,x)/W_\alpha^2(x)\leqslant C_5a_nn^{-2/3},$$

for $a_{n(1-2n^{-2/3})} \leqslant |x| \leqslant a_n(1+Ln^{-2/3})$. Together with (3.4), this last inequality shows that

$$\lambda_n(W_\alpha^2, x) \leqslant C_6 \frac{a_n}{n} W_\alpha^2(x) \left(\max\left\{n^{-2/3}, 1-\frac{|x|}{a_n}\right\}\right)^{1/2},$$

for $\varepsilon a_n \leqslant |x| \leqslant a_n(1+Ln^{-2/3})$. The corresponding lower bound for $\lambda_n(W_\alpha^2, x)$ is a special case of Theorem 1.7 in [10]. ▮

4. Upper Bounds for Christoffel Functions, II

In this section, we obtain upper bounds for $\lambda_n(W_\alpha^2, x)$ using the method of Freud, Giroux, and Rahman [7], for the range $|x| \leqslant \varepsilon a_n$, some suitable $\varepsilon > 0$, and hence deduce Theorem 1.1(a) for this range. We note that an alternative derivation of this upper bound may be based on the asymptotics for Christoffel functions given in Theorem I.3 for [11, p. 185], more specifically (I.15) there. Although this would shorten this section, we follow the method of [7], since this avoids using the "deep" results in [11].

We use the canonical product

$$P_\beta(z) := \prod_{n=1}^{\infty} (1 + z/n^{1/\beta}), \qquad z \in \mathbb{C}, \quad 0 < \beta < 1, \tag{4.1}$$

and the asymptotics for P_β in [1]. It is known [1, p. 497, Thm. 1] that

$$\log P_\beta(z) = \frac{\pi}{\sin \pi\beta} z^\beta - \frac{1}{2}\log z - \frac{1}{2\beta}\log(2\pi) + \mathscr{E}(z), \tag{4.2}$$

where $\mathscr{E}(z) \to 0$ as $z \to \infty$, uniformly in the sector $\{z : |\arg z| < \pi - \delta\}$, for any fixed $\delta \in (0, \pi)$. We shall set

$$\tau_\alpha := \left(\frac{\pi}{\sin \pi\alpha/2}\right)^{-1/\alpha}; \tag{4.3}$$

and for $0 < \alpha \leqslant 1$, we set

$$\begin{aligned} H_\alpha(z) &:= (1 + (\tau_\alpha z)^2)\, P_{\alpha/2}^2((\tau_\alpha z)^2) \\ &= (1 + (\tau_\alpha z)^2) \prod_{n=1}^{\infty} (1 + (\tau_\alpha z/n^{1/\alpha})^2)^2. \end{aligned} \tag{4.4}$$

Lemma 4.1. *H_α is an even entire function with non-negative Maclaurin series coefficients, such that*

$$H_\alpha(x)\, W_\alpha^2(x) \sim 1, \qquad x \in \mathbb{R}. \tag{4.5}$$

Proof. For large $|x|$, the $\sim$ in (4.5) follows easily from (4.2). For small $|x|$, this follows as H_α and W_α^2 are positive and continuous in $\mathbb{R}$. ∎

LEMMA 4.2. *Let $0<\alpha\leqslant 1$. Let $S_m\in\mathscr{P}_m$ denote the mth partial sum of the Maclaurin series of*

$$G(z):=P_{\alpha/2}((\tau_\alpha z)^2). \tag{4.6}$$

Then

(a) *S_m is an even polynomial with non-negative Maclaurin series coefficients.*

(b) *Let $\varepsilon>0$. Then there exists $\delta>0$ such that for $m\geqslant 1$,*

$$|G(z)-S_m(z)|\leqslant\varepsilon^m,\qquad |z|\leqslant(\delta m)^{1/\alpha}, \tag{4.7}$$

and

$$|S_m(x)/G(x)-1|\leqslant\tfrac{1}{2},\qquad |x|\leqslant(\delta m)^{1/\alpha}. \tag{4.8}$$

(c) *Let $\sigma\geqslant 1$. There exists $\eta>0$ such that for $m\geqslant 1$, the only zeros of $S_m(z)$ inside the rectangle with vertices $\pm\sigma\langle\eta m\rangle^{1/\alpha}\pm i\langle\eta m\rangle^{1/\alpha}$ lie on the imaginary axis. These zeros are simple, and have the form $\pm iy_j/\tau_\alpha$, where*

$$y_j^\alpha\in(j-\tfrac{1}{2},j+\tfrac{1}{2}),\qquad 1\leqslant j\leqslant\langle\eta m\rangle^{1/\alpha}. \tag{4.9}$$

Proof. (a) This is immediate.

(b) This is an easy consequence of the contour integral error formula for $G-S_m$, and (4.1), (4.2).

(c) We use Rouché's theorem, applied to $G(z)$ and $S_m(z)$. To this end, we find suitable lower bounds for $|G(z)|$. Suppose that $l\geqslant 0$ and

$$\tau_\alpha z=x+i(l+\tfrac{1}{2})^{1/\alpha},\qquad x\in\mathbb{R}.$$

Note the inequality

$$|1+\zeta^2|^2\geqslant|1-(\operatorname{Im}\zeta)^2|^2,\qquad \zeta\in\mathbb{C}.$$

Then for $j\geqslant 1$, this inequality yields

$$\left|1+\left[\frac{\tau_\alpha z}{j^{1/\alpha}}\right]^2\right|^2\geqslant\left(1-\left[\frac{l+\frac{1}{2}}{j}\right]^{2/\alpha}\right)^2.$$

So for such z, we see that

$$|G(z)| \geqslant \left|G\left(i\left(\frac{(l+\frac{1}{2})^{1/\alpha}}{\tau_\alpha}\right)\right)\right| = (-1)^l G\left(i\left(\frac{(l+\frac{1}{2})^{1/\alpha}}{\tau_\alpha}\right)\right)$$

$$= \prod_{j=1}^{2l} \left|1 - \left[\frac{l+\frac{1}{2}}{j}\right]^{2/\alpha}\right| \prod_{j=2l+1}^{\infty} \left|1 - \left[\frac{l+\frac{1}{2}}{j}\right]^{2/\alpha}\right|$$

$$=: \Pi_1 \cdot \Pi_2.$$

Here, as $l \to \infty$, we can write

$$\Pi_1 = \exp\left(\sum_{j=1}^{2l} \log\left|1 - \left[\frac{1+1/2l}{j/l}\right]^{2/\alpha}\right|\right)$$

$$= \exp\left(l\left[\int_0^2 \log\left|1 - \left[\frac{1}{x}\right]^{2/\alpha}\right| dx + o(1)\right]\right)$$

$$\geqslant \exp(-C_1 l).$$

Next,

$$\Pi_2 = \exp\left(\sum_{j=2l+1}^{\infty} \log\left|1 - \left[\frac{l+\frac{1}{2}}{j}\right]^{2/\alpha}\right|\right)$$

$$\geqslant \exp\left(-2 \sum_{j=2l+1}^{\infty} \left[\frac{l+\frac{1}{2}}{j}\right]^{2/\alpha}\right) \geqslant \exp(-C_2 l).$$

Here we have used the inequality

$$\log(1-x) \geqslant -2x, \qquad x \in (0, \tfrac{1}{2}].$$

So

$$|G(z)| \geqslant \exp(-C_3 l), \qquad z = \frac{x}{\tau_\alpha} + \frac{i(l+\frac{1}{2})^{1/\alpha}}{\tau_\alpha}, x \in \mathbb{R}.$$

Moreover, using (4.2), we see that

$$|G(z)| \geqslant C, \qquad z = \sigma\frac{(l+\frac{1}{2})^{1/\alpha}}{\tau_\alpha} + i\frac{y}{\tau_\alpha}, |y| \leqslant \left(l+\frac{1}{2}\right)^{1/\alpha}.$$

Let $\mathscr{R}_l$ denote the rectangular contour with vertices

$$\pm\sigma\frac{(l+\frac{1}{2})^{1/\alpha}}{\tau_\alpha} \pm i\frac{(l+\frac{1}{2})^{1/\alpha}}{\tau_\alpha}.$$

We have shown that

$$|G(z)| \geqslant \exp(-C_3 l), \qquad z \in \mathscr{R}_l.$$

Let $0 < \varepsilon < \exp(-C_3)$ and δ be as in (b) of this lemma. Choose $\eta > 0$ so small that if $l := \langle \eta m \rangle$, $\mathscr{R}_l \subset \{z : |z| \leqslant (\delta m)^{1/\alpha}\}$. Then by (b) of this lemma, for z inside and on $\mathscr{R}_l$,

$$|G(z) - S_m(z)| \leqslant \varepsilon^m < e^{-C_3 m} < e^{-C_3 l} < |G(z)|.$$

By Rouché's theorem, $S_m(z)$ has the same total multiplicity of zeros inside $\mathscr{R}_l$ as G. Now G has its zeros in $\mathscr{R}_l$, all simple, precisely, at $\pm i j^{1/\alpha}/\tau_\alpha$, $1 \leqslant j \leqslant l$. We showed above that

$$(-1)^j G\left(i\left(\frac{(j+\frac{1}{2})^{1/\alpha}}{\tau_\alpha}\right)\right) \geqslant \exp(-C_3 l), \qquad 0 \leqslant |j| \leqslant l.$$

(For $j = 0$, and l large enough, this is trivial.) As $\varepsilon < \exp(-C_3)$, we have

$$(-1)^j S_m\left(i\left(\frac{(j+\frac{1}{2})^{1/\alpha}}{\tau_\alpha}\right)\right) > 0, \qquad 0 \leqslant |j| \leqslant l.$$

So, S_m has zeros $\pm i y_j/\tau_\alpha$, where for $1 \leqslant j \leqslant l$, $y_j^\alpha \in [j - \frac{1}{2}, j + \frac{1}{2}]$. ∎

Next, we consider the polynomial

$$Q_n(z) := (1 + (\tau_\alpha z)^2)\, S_{n-2}^2(z) \in \mathscr{P}_{2n-2}, \qquad n \geqslant 1. \tag{4.10}$$

LEMMA 4.3. (a) $\exists\, C_1, C_2 > 0$ *such that*

$$\sup_{x \in \mathbb{R}} Q_n(x)\, W_\alpha^2(x) \leqslant C_1, \qquad n \geqslant 1; \tag{4.11}$$

$$Q_n(x)\, W_\alpha^2(x) \sim 1, \qquad |x| \leqslant (C_2 n)^{1/\alpha},\ n \geqslant 1. \tag{4.12}$$

(b) *We can write*

$$Q_n\left(a_n \frac{1}{2}\left[z + \frac{1}{z}\right]\right) = h_n(z)\, \overline{h_n(1/\bar{z})}, \qquad z \in \mathbb{C} \setminus \{0\}, \tag{4.13}$$

where $h_n \in \mathscr{P}_{n-1}$, $h_n(0) > 0$ *and has all its zeros in* $\{z : |z| > 1\}$. *Moreover,* h_n *has double zeros at*

$$\pm i\xi_k = \pm i\left(\frac{y_k}{\tau_\alpha a_n} + \sqrt{1 + \left(\frac{y_k}{\tau_\alpha a_n}\right)^2}\right), \qquad 1 \leqslant k \leqslant \langle \eta n \rangle, \tag{4.14}$$

where the $\{y_k\}$ *are as in the previous lemma* (*for* $m=n-2$); h_n *has simple zeros at*

$$\pm i\xi_0 = \pm i\left(\frac{1}{\tau_\alpha a_n} + \sqrt{1+\left(\frac{1}{\tau_\alpha a_n}\right)^2}\right). \tag{4.15}$$

All other zeros of h_n *lie in* $\{z : |z| \geqslant 1 + C\}$, *some* C *independent of* n.

Proof. (a) This follows easily from (4.5), (4.8), and (4.10).

(b) We note that for $z = e^{i\theta}$,

$$Q_n\left(a_n\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]\right) = Q_n(a_n \cos\theta) > 0.$$

Then the decomposition (4.13) is well known [23, p. 3]. We see that

$$a_n\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right] = \pm i y_k/\tau_\alpha$$

is equivalent to (for $|z| > 1$)

$$z = \pm i\xi_k = \pm i\left(\frac{y_k}{\tau_\alpha a_n} + \sqrt{1+\left(\frac{y_k}{\tau_\alpha a_n}\right)^2}\right).$$

It is easily seen that the double zero of Q_n leads to a double zero of h_n. Similarly, we may discuss the simple zeros of Q_n at $\pm i/\tau_\alpha$.

Finally, the map

$$w = \frac{1}{2}\left(z+\frac{1}{z}\right) \Leftrightarrow z = w + \sqrt{w^2-1}$$

maps circles with centre 0 in the z-plane onto ellipses with foci at ± 1 in the w-plane. For suitable σ large enough, we can fit an ellipse with foci at ± 1 inside the rectangle with vertices at $\pm\sigma\langle\eta n\rangle^{1/\alpha}/a_n \pm i\langle\eta n\rangle^{1/\alpha}/a_n$, and having the same intercepts on the real and imaginary axes. Here

$$\langle\eta n\rangle^{1/\alpha}/a_n \geqslant C_1 > 0.$$

As the only zeros of $Q_n(a_n[\frac{1}{2}(z+1/z)])$ inside this ellipse are the simple/double zeros listed above, all other zeros outside this ellipse correspond to zeros of h_n in $\{z : |z| \geqslant 1 + C\}$, with $C > 0$ independent of n. ∎

Proof of Theorem 1.1(a) *for the range* $|x| < \varepsilon a_n$. Now by the infinite–finite range inequality Lemma 3.1, and then by Lemma 4.3, we have for $|x| \leqslant (C_1 n)^{1/\alpha}$

$$\lambda_{n+1}(W_\alpha^2, x)/W_\alpha^2(x)$$

$$\leqslant C_2 \inf_{P\in\mathscr{P}_n} \int_{-a_n}^{a_n} (PW_\alpha)^2(t)\,dt/(PW_\alpha)^2(x)$$

$$\leqslant C_3 \inf_{P\in\mathscr{P}_n} \int_{-a_n}^{a_n} P^2(t)\,Q_n^{-1}(t)\,dt/(P^2(x)\,Q_n^{-1}(x))$$

$$\leqslant C_4 a_n \inf_{R\in\mathscr{P}_n} \int_{-1}^{1} R^2(t)(1-t^2)^{-1/2}$$

$$\times Q_n^{-1}(a_n t)\,dt/(R^2(x/a_n)\,Q_n^{-1}(x))$$

$$= C_4 a_n \lambda_{n+1}(w_n, x/a_n)\,Q_n(x), \tag{4.16}$$

where

$$w_n(x) := (1-x^2)^{-1/2}\,Q_n^{-1}(a_n x). \tag{4.17}$$

Recall the definition of h_n in the previous lemma. It is known [23, p. 322; 7, p.363] that if

$$x/a_n = \cos\phi; \qquad \phi\in[0,\pi]; \qquad z = e^{i\phi}, \tag{4.18}$$

then

$$\pi\lambda_{n+1}^{-1}(w_n, x)(1-x^2)^{1/2}\,w_n(x)$$

$$= n + \tfrac{1}{2} - \operatorname{Re}(zh_n'(z)/h_n(z)) + (2\sin\phi)^{-1}\operatorname{Im}(z^{2n+1}\overline{h_n(z)}/h_n(z))$$

$$= n - \operatorname{Re}(zh_n'(z)/h_n(z)) + O(1), \tag{4.19}$$

for $|x|/a_n \leqslant \frac{1}{2}$, say. We now estimate the second term in the right-hand side of (4.19). We can write (recall (4.14), (4.15))

$$h_n(z) = c(z^2-(i\xi_0)^2)\prod_{j=1}^{\langle\eta n\rangle}(z^2-(i\xi_j)^2)^2\,g_n(z),$$

where g_n is a monic polynomial of degree $<n$ with all its zeros in $\{z : |z|\geqslant 1+C\}$. Let us suppose that in (4.18),

$$\phi\in\left[0, \frac{\pi}{2}\right],$$

so that $x \geqslant 0$. We see that

$$\begin{aligned} -zh_n'(z)/h_n(z) &= -z\left[\frac{1}{z-i\xi_0}+\frac{1}{z+i\xi_0}\right] \\ &\quad -2z\sum_{j=1}^{\langle \eta n\rangle}\left[\frac{1}{z-i\xi_j}+\frac{1}{z+i\xi_j}\right]-z\sum_{|\xi_j|\geqslant 1+C}\frac{1}{z-i\xi_j} \\ &= -2z\sum_{j=0}^{\langle \eta n\rangle}{}^{\#}\left[\frac{1}{z-i\xi_j}+\frac{1}{z+i\xi_j}\right]+O(n), \end{aligned}$$

uniformly for such x. Here and in the sequel # means that the term for $j=0$ is multiplied by $\frac{1}{2}$. Here

$$\begin{aligned} \operatorname{Re}\left\{-z\left[\frac{1}{z-i\xi_j}+\frac{1}{z+i\xi_j}\right]\right\} &= \frac{\xi_j\sin\phi-1}{1-2\xi_j\sin\phi+\xi_j^2}+\frac{-\xi_j\sin\phi-1}{1+2\xi_j\sin\phi+\xi_j^2} \\ &= \frac{\xi_j\sin\phi-1}{1-2\xi_j\sin\phi+\xi_j^2}+O(1), \end{aligned}$$

as $\sin\phi\geqslant 0$. So

$$\begin{aligned} \operatorname{Re}\{-zh_n'(z)/h_n(z)\} &= 2\sum_{j=0}^{\langle \eta n\rangle}{}^{\#}\frac{\xi_j\sin\phi-1}{1-2\xi_j\sin\phi+\xi_j^2}+O(n) \\ &= 2\sin\phi\sum\nolimits_1+2(\sin\phi-1)\sum\nolimits_2+O(n), \qquad (4.20) \end{aligned}$$

where

$$\begin{aligned} \sum\nolimits_1 &:= \sum_{j=0}^{\langle \eta n\rangle}{}^{\#}\frac{\xi_j-1}{1-2\xi_j\sin\phi+\xi_j^2}; \\ \sum\nolimits_2 &:= \sum_{j=0}^{\langle \eta n\rangle}{}^{\#}\frac{1}{1-2\xi_j\sin\phi+\xi_j^2}. \end{aligned}$$

Now

$$1-\sin\phi\sim\cos^2\phi=(x/a_n)^2\sim(x/n^{1/\alpha})^2.$$

Also, since

$$(w+\sqrt{1+w^2})-1\sim w, \qquad w\in(-\tfrac{1}{2},\tfrac{1}{2}),$$

we have (see (4.9), (4.14))

$$\xi_j-1\sim\frac{y_j}{\tau_\alpha a_n}\sim\left(\frac{j}{n}\right)^{1/\alpha}, \qquad 1\leqslant j\leqslant\langle \eta n\rangle.$$

The $\sim$ holds uniformly in j and n. Further, by (4.15),

$$\xi_0 - 1 \sim \frac{1}{a_n} \sim \left(\frac{1}{n}\right)^{1/\alpha}.$$

Then

$$\begin{aligned}\sum\nolimits_1 &= \sum_{j=0}^{\langle \eta n \rangle}{}^{\#} \frac{\xi_j - 1}{(\xi_j - 1)^2 + 2\xi_j(1 - \sin\phi)} \\ &\sim \sum_{j=1}^{\langle \eta n \rangle} \frac{(j/n)^{1/\alpha}}{(j/n)^{2/\alpha} + (x/n^{1/\alpha})^2} \\ &\sim n^{1/\alpha} \int_0^{\eta n} \frac{u^{1/\alpha}}{u^{2/\alpha} + x^{*2}}\, du,\end{aligned}$$

where $x^* := \max\{1, x\}$. The substitution $u = (x^*)^\alpha s$ yields

$$\begin{aligned}\sum\nolimits_1 &\sim n^{1/\alpha}(x^*)^{\alpha - 1} \int_0^{\eta n/(x^*)^\alpha} \frac{s^{1/\alpha}}{s^{2/\alpha} + 1}\, ds \\ &\sim n^{1/\alpha} \begin{cases} \log(n/x^*), & \alpha = 1, \\ (x^*)^{\alpha - 1}, & \alpha < 1, \end{cases}\end{aligned} \tag{4.21}$$

if $0 \leqslant x \leqslant \varepsilon a_n$, some small enough $\varepsilon > 0$. Similarly,

$$\begin{aligned}\sum\nolimits_2 &= \sum_{j=0}^{\langle \eta n \rangle}{}^{\#} \frac{1}{(\xi_j - 1)^2 + 2\xi_j(1 - \sin\phi)} \\ &\sim \sum_{j=1}^{\langle \eta n \rangle} \frac{1}{(j/n)^{2/\alpha} + (x/n^{1/\alpha})^2} \\ &\sim n^{2/\alpha} \int_0^{\eta n} \frac{1}{u^{2/\alpha} + (x^*)^2}\, dx \\ &= n^{2/\alpha}(x^*)^{\alpha - 2} \int_0^{\eta n/(x^*)^\alpha} \frac{ds}{s^{2/\alpha} + 1} \sim n^{2/\alpha}(x^*)^{\alpha - 2}.\end{aligned} \tag{4.22}$$

So combining (4.20), (4.21), and (4.22), we have for $0 \leqslant x \leqslant \varepsilon a_n$,

$$\begin{aligned}\operatorname{Re}\{-zh_n'(z)/h_n(z)\} &\geqslant \sum\nolimits_1 - 2\left(\frac{x}{a_n}\right)^2 \sum\nolimits_2 + O(n) \\ &\geqslant C_1 n^{1/\alpha} \left.\begin{cases} \log(n/x^*), & \alpha = 1 \\ (x^*)^{\alpha - 1}, & \alpha < 1 \end{cases}\right\} - C_2(x^*)^\alpha + O(n) \\ &\geqslant C_3 a_n \begin{cases} \log(n/x^*), & \alpha = 1, \\ (x^*)^{\alpha - 1}, & \alpha < 1, \end{cases}\end{aligned}$$

if ε is small enough. Substituting this estimate into (4.19) gives

$$\pi\lambda_{n+1}^{-1}(w_n, x/a_n)\, Q_n^{-1}(x) \geqslant C_4 a_n \begin{cases} \log(n/x^*), & \alpha = 1 \\ (x^*)^{\alpha-1}, & \alpha < 1. \end{cases}$$

Finally, (4.16) gives for $x \in [0, \varepsilon a_n]$

$$\lambda_{n+1}(W_\alpha^2, x)/W_\alpha^2(x) \leqslant C_5 \begin{cases} (\log(n/x^*))^{-1}, & \alpha = 1, \\ (x^*)^{1-\alpha}, & \alpha < 1. \end{cases}$$

The corresponding lower bound is a special case of Theorem 1.7 in [10]. ▮

5. Zeros of Orthogonal Polynomials

We begin with the

Proof of Corollary 1.2(a). We use the well known formula [6]

$$x_{1n} = \sup_{\substack{P \in \mathscr{P}_{2n-2} \\ P \geqslant 0}} \int_{-\infty}^{\infty} xP(x)\, W_\alpha^2(x)\, dx \Big/ \int_{-\infty}^{\infty} P(x)\, W_\alpha^2(x)\, dx,$$

which is an easy consequence of the Gauss quadrature formula. Then

$$a_n - x_{1n} = \inf_{\substack{P \in \mathscr{P}_{2n-2} \\ P \geqslant 0}} \int_{-\infty}^{\infty} (a_n - x)\, P(x)\, W_\alpha^2(x)\, dx \Big/ \int_{-\infty}^{\infty} P(x)\, W_\alpha^2(x)\, dx.$$

Since a_{2n} for W_α^2 is a_n for W_α, we can use Lemma 3.1 to deduce that

$$a_n - x_{1n} \leqslant C \inf_{\substack{P \in \mathscr{P}_{2n-2} \\ P \geqslant 0}} \int_{-a_n}^{a_n} (a_n - x)\, P(x)\, W_\alpha^2(x)\, dx \Big/ \int_{-a_n}^{a_n} P(x)\, W_\alpha^2(x)\, dx. \tag{5.1}$$

Now we set

$$m := \langle n^{1/3}/2 \rangle,$$

and

$$P(x) := \lambda_{n-2m}^{-1}(W_\alpha^2, x)\, l_{1m}^4(a_n^{-1} x),$$

where l_{1m} is the fundamental polynomial of Lagrange interpolation of degree m for interpolation at the zeros of the Chebyshev polynomial

$T_m(x)$, corresponding to the largest zero of $T_m(x)$. Note that by the first part of Theorem 1.1,

$$\lambda_{n-2m}^{-1}(W_\alpha^2, x)\, W_\alpha^2(x) \sim \Lambda_n(x)^{-1},$$

for $|x| \leqslant a_n$, since (cf. (1.5))

$$a_n/a_{n-m} = 1 + O(n^{-2/3}).$$

Then we obtain from (5.1) and this estimate,

$$a_n - x_{1n} \leqslant C_1 a_n \int_{-1}^{1} (1-s)\, l_{1m}^4(s)\, \Lambda_n^{-1}(a_n s)\, ds \Big/ \int_{-1}^{1} l_{1m}^4(s)\, \Lambda_n^{-1}(a_n s)\, ds. \tag{5.2}$$

Now it is known that for some C_1 and C_2 (see, for example, [10, p. 531])

$$|l_{1m}(s)| \leqslant \frac{C_1}{m^2 |s - x_{1m}^*|}; \qquad |l_{1m}(s)| \leqslant C_1,\ s \in [-1, 1];\ m \geqslant 1.$$

$$l_{1m}(s) \geqslant \frac{1}{2}, \qquad |s - x_{1m}^*| \leqslant C_2 m^{-2}.$$

Here $x_{1m}^* := \cos(\pi/2m)$ denotes the largest zero of $T_m(x)$. We turn to the estimation of the integrals in (5.2). We write for $k = 0, 1$,

$$\begin{aligned}
&\int_{-1}^{1} (1-s)^k\, l_{1m}^4(s)\, \Lambda_n^{-1}(a_n s)\, ds \\
&= \left[\int_{-1}^{1/2} + \int_{1/2}^{x_{1m}^* - C_2 m^{-2}} + \int_{x_{1m}^* - C_2 m^{-2}}^{x_{1m}^* + C_2 m^{-2}} + \int_{x_{1m}^* + C_2 m^{-2}}^{1}\right] \\
&\quad \times (1-s)^k\, l_{1m}^4(s)\, \Lambda_n^{-1}(a_n s)\, ds \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

The estimates for l_{1m} and (1.7), (1.8) readily yield

$$I_1 \leqslant C_3 m^{-8} n^{1-1/\alpha} \log n;$$

$$I_2 \leqslant C_4 m^{-8} n^{1-1/\alpha} \int_{1/2}^{x_{1m}^* - C_2 m^{-2}} |s - x_{1m}^*|^{-4} (1-s)^{k+1/2}\, ds$$

(recall that $n^{-2/3} \sim m^{-2}$)

$$\leqslant C_5 m^{-8} n^{1-1/\alpha} \int_{C_2 m^{-2}}^{1} u^{-4}(u+1-x_{1m}^*)^{k+1/2}\, du$$

$$\leqslant C_6 n^{1-1/\alpha} m^{-2k-3} \int_{C_2}^{\infty} w^{-4}(w+1)^{k+1/2}\, dw$$

$$\leqslant C_7 n^{1-1/\alpha} m^{-2k-3}.$$

Next,

$$I_3 \sim n^{1-1/\alpha} \int_{x_{1m}^*-C_2 m^{-2}}^{x_{1m}^*+C_2 m^{-2}} (1-s)^{k+1/2}\, ds$$
$$\sim n^{1-1/\alpha}(m^{-2})^{k+3/2} = n^{1-1/\alpha} m^{-2k-3}.$$

Finally, we similarly deduce that

$$I_4 \leqslant C_8 n^{1-1/\alpha} m^{-2k-3}.$$

So, combining these estimates, we have shown that for $k=0, 1$,

$$\int_{-1}^{1} (1-s)^k\, l_{1m}^4(s)\, \Lambda_n^{-1}(a_n s)\, ds \sim n^{1-1/\alpha} m^{-2k-3}.$$

Then from (5.2), we obtain

$$a_n - x_{1n} \leqslant C_9 a_n m^{-2} \sim a_n n^{-2/3}.$$

For the converse inequality, we note that if $K>0$ is large enough, then for n large enough and $P \in \mathscr{P}_n$, we have

$$\int_{|x| \geqslant a_n(1+Kn^{-2/3})} |RW_\alpha^2|\,(x)\, dx \leqslant \frac{1}{2} \int_{|x| \leqslant a_n(1+Kn^{-2/3})} |RW_\alpha^2|\,(x)\, dx.$$

This follows directly from Theorem 1.8 in [10, p. 469] and (7.14), (10.6) in [10, p. 486, p. 513]. Hence for $P \in \mathscr{P}_{2n-2}$, with $P \geqslant 0$,

$$\int_{-\infty}^{\infty} [a_n(1+Kn^{-2/3}) - x](PW_\alpha^2)(x)\, dx$$
$$\geqslant \frac{1}{2} \int_{|x| \leqslant a_n(1+Kn^{-2/3})} [a_n(1+Kn^{-2/3}) - x](PW_\alpha^2)(x)\, dx$$
$$\geqslant 0.$$

Then

$$a_n(1+Kn^{-2/3}) - x_{1n}$$
$$= \inf_{\substack{P \in \mathscr{P}_{2n-2} \\ P \geqslant 0}} \int_{-\infty}^{\infty} [a_n(1+Kn^{-2/3}) - x](PW_\alpha^2)(x)\, dx \Big/ \int_{-\infty}^{\infty} (PW_\alpha^2)(x)\, dx$$
$$\geqslant 0. \quad \blacksquare$$

Proof of Corollary 1.2(b). Let H_α be the entire function defined at (4.4) and recall (4.5). Also, define the Christoffel numbers

$$\lambda_{jn} := \lambda_n(W_\alpha^2, x_{jn}), \qquad 1 \leqslant j \leqslant n.$$

Now we use the Posse–Markov–Stieltjes inequalities in the form given in [8, p. 89]: For $2 \leqslant j \leqslant n-1$,

$$\begin{aligned} \lambda_{jn} H_\alpha(x_{jn}) &= \frac{1}{2}\left[\sum_{k\,:\,|x_{kn}| < |x_{j-1,n}|} \lambda_{kn} H_\alpha(x_{kn}) - \sum_{k\,:\,|x_{kn}| < |x_{jn}|} \lambda_{kn} H_\alpha(x_{kn})\right] \\ & \frac{1}{2}\left[\int_{-x_{j-1,n}}^{x_{j-1,n}} - \int_{-x_{j+1,n}}^{x_{j+1,n}}\right] H_\alpha(t)\, W_\alpha^2(t)\, dt \\ &= \int_{x_{j+1,n}}^{x_{j-1,n}} H_\alpha(t)\, W_\alpha^2(t)\, dt. \end{aligned} \tag{5.3}$$

Moreover, we similarly obtain

$$\begin{aligned} &\lambda_{jn} H_\alpha(x_{jn}) + \lambda_{j+1,n} H_\alpha(x_{j+1,n}) \\ &\quad = \frac{1}{2}\left[\sum_{k\,:\,|x_{kn}| < |x_{j-1,n}|} \lambda_{kn} H_\alpha(x_{kn}) - \sum_{k\,:\,|x_{kn}| < |x_{j+1,n}|} \lambda_{kn} H_\alpha(x_{kn})\right] \\ &\quad \geqslant \frac{1}{2}\left[\int_{-x_{jn}}^{x_{jn}} - \int_{-x_{j+1,n}}^{x_{j+1,n}}\right] H_\alpha(t)\, W_\alpha^2(t)\, dt \\ &\quad = \int_{x_{j+1,n}}^{x_{jn}} H_\alpha(t)\, W_\alpha^2(t)\, dt. \end{aligned} \tag{5.4}$$

Then (4.5), (5.3), and (5.4) yield

$$\lambda_{jn} W_\alpha^{-2}(x_{jn}) \leqslant C_1(x_{j-1,n} - x_{j+1,n})$$

and

$$\lambda_{jn} W_\alpha^{-2}(x_{jn}) + \lambda_{j+1,n} W_\alpha^{-2}(x_{j+1,n}) \geqslant C_2(x_{jn} - x_{j+1,n}).$$

Then Theorem 1.1 enables us to conclude that

$$x_{j-1,n} - x_{j+1,n} \geqslant C_3 \Lambda_n(x_{jn})$$

and

$$x_{jn} - x_{j+1,n} \leqslant C_4\{\Lambda_n(x_{jn}) + \Lambda_n(x_{j+1,n})\}. \tag{5.5}$$

The proof will be complete if we can show that uniformly for $2 \leqslant j \leqslant n-1$,

$$\Lambda_n(x_{jn}) \sim \Lambda_n(x_{j+1,n}). \tag{5.6}$$

Now, if $x_{j+1,n} \geqslant 0$, and $0 \leqslant x_{jn} \leqslant a_n/2$, then for $0 < \alpha < 1$, (1.7) and (5.5) give

$$1 \leqslant \frac{1+x_{jn}}{1+x_{j+1,n}} \leqslant 1 + C_4 \frac{\Lambda_n(x_{jn}) + \Lambda_n(x_{j+1,n})}{1+x_{j+1,n}}$$
$$\leqslant C_5 + C_4(1+x_{jn})^{-\alpha} \frac{1+x_{jn}}{1+x_{j+1,n}}.$$

Then (5.6) follows. If $\alpha = 1$, (1.7) and (5.5) show that

$$x_{jn} - x_{j+1,n} \leqslant C_6,$$

and then again, we obtain (5.6).

Next, if $x_{j+1,n} \geqslant 0$, and $a_n/2 \leqslant x_{jn} \leqslant a_n(1 - n^{-2/3})$,

$$1 \leqslant \frac{1 - x_{j+1,n}/a_n}{1 - x_{jn}/a_n} = 1 + \frac{x_{jn} - x_{j+1,n}}{a_n(1 - x_{jn}/a_n)}$$
$$= 1 + O\left(\frac{1}{n}(1 - x_{jn}/a_n)^{-3/2}\right) = O(1).$$

On the other hand if $x_{jn} \geqslant a_n(1 - n^{-2/3})$, then (5.5) and Corollary 1.2(a) yield

$$\left|1 - \frac{x_{j+1,n}}{a_n}\right| \leqslant \left|1 - \frac{x_{jn}}{a_n}\right| + \frac{x_{jn} - x_{j+1,n}}{a_n}$$
$$\leqslant C_7 n^{-2/3} + C_7 \frac{1}{n} \max\left\{n^{-2/3}, 1 - \frac{x_{j+1,n}}{a_n}\right\}^{-1/2}$$
$$\leqslant C_8 n^{-2/3}.$$

We have thus shown that, for $x_{jn} \geqslant a_n/2$,

$$\max\left\{n^{-2/3}, 1 - \frac{|x_{jn}|}{a_n}\right\} \sim \max\left\{n^{-2/3}, 1 - \frac{|x_{j+1,n}|}{a_n}\right\}. \tag{5.7}$$

Hence we have (5.6) uniformly in j and n such that $x_{j+1,n} \geqslant 0$. The proof of (5.6) for the remaining cases is similar. ∎

6. Bounds for Orthogonal Polynomials

In this section, we prove Corollaries 1.3 and 1.4. Our method for finding upper bounds for orthogonal polynomials is similar to that in [10], but we have been unable to provide complete results as in [10] because of the difficulty of estimating a certain function.

We shall need the *Christoffel–Darboux* formula

$$K_n(x,t) := K_n(W_\alpha^2, x, t) := \sum_{j=0}^{n-1} p_j(x)\, p_j(t)$$
$$= \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)\, p_{n-1}(t) - p_{n-1}(x)\, p_n(t)}{x-t} \tag{6.1}$$

(Recall that we abbreviate $p_n(x) = p_n(W_\alpha^2, x)$.) From this it follows by setting $x = t = x_{jn}$ that

$$\lambda_{jn}^{-1} = \frac{\gamma_{n-1}}{\gamma_n} p_n'(x_{jn})\, p_{n-1}(x_{jn}). \tag{6.2}$$

We define, as in [10, 15], with $Q(x) := |x|^\alpha$,

$$A_n(x) := 2\frac{\gamma_{n-1}}{\gamma_n} \int_{-\infty}^{\infty} p_n^2(t)\, W_\alpha^2(t) \frac{Q'(x) - Q'(t)}{x-t}\, dt$$
$$= 2\frac{\gamma_{n-1}}{\gamma_n} \int_0^{\infty} p_n^2(t)\, W_\alpha^2(t)\, \hat{Q}(x,t)\, dt, \tag{6.3}$$

where if $x, t \geq 0$,

$$\hat{Q}(x,t) := \frac{xQ'(x) - tQ'(t)}{x^2 - t^2} = \alpha \frac{x^\alpha - t^\alpha}{x^2 - t^2}. \tag{6.4}$$

It is known that [15, Thm. 3,2]

$$p_n'(x_{jn}) = A_n(x_{jn})\, p_{n-1}(x_{jn}) \tag{6.5}$$

and hence (6.2) becomes

$$\lambda_{jn}^{-1} = \frac{\gamma_{n-1}}{\gamma_n} A_n(x_{jn})\, p_{n-1}^2(x_{jn}). \tag{6.6}$$

Estimation of $A_n(x)$ plays a major role:

LEMMA 6.1. *Uniformly for $n \geq 1$ and $0 < x \leq 2a_n$,*

$$A_n(x)\Big/\frac{\gamma_{n-1}}{\gamma_n} \sim x^{\alpha-2} \int_0^{\min\{x, a_n\}} (p_n W_\alpha)^2(t)\, dt + \int_{\min\{x, a_n\}}^{a_n} (p_n W_\alpha)^2(t)\, t^{\alpha-2}\, dt. \tag{6.7}$$

Moreover,

$$C_1 a_n^{\alpha-2} \leq A_n(x)\Big/\frac{\gamma_{n-1}}{\gamma_n} \leq C_2 x^{\alpha-2}. \tag{6.8}$$

Proof. It is readily seen that $\hat{Q}(x,t)$ defined by (6.4) satisfies

$$\hat{Q}(x,t) \sim \max\{t, x\}^{\alpha-2} \quad \text{uniformly for} \quad t, x \in (0, \infty). \tag{6.9}$$

Then we see that for $x \in (0, 2a_n]$,

$$\int_0^{a_n} (p_n W_\alpha)^2 (t)\, \hat{Q}(x, t)\, dt \sim x^{\alpha-2} \int_0^{\min\{x, a_n\}} (p_n W_\alpha)^2 (t)\, dt$$

$$+ \int_{\min\{x, a_n\}}^{a_n} (p_n W_\alpha)^2 (t)\, t^{\alpha-2}\, dt. \tag{6.10}$$

Note that a lower bound for the last right-hand side is

$$(2a_n)^{\alpha-2} \int_0^{a_n} (p_n W_\alpha)^2 (t)\, dt \sim a_n^{\alpha-2} \int_{-\infty}^{\infty} (p_n W_\alpha)^2 (t)\, dt = a_n^{\alpha-2},$$

in view of the evenness of $(p_n W_\alpha)^2$ and the infinite–finite range inequality Lemma 3.1. Next,

$$\int_{a_n}^{\infty} (p_n W_\alpha)^2 (t)\, \hat{Q}(x, t)\, dt \sim \int_{a_n}^{\infty} (p_n W_\alpha)^2 (t)\, t^{\alpha-2}\, dt \leqslant a_n^{\alpha-2}.$$

Then (6.7) follows from (6.3), (6.10), and this last inequality. Finally, (6.8) is immediate. ∎

Proof of Corollary 1.4. First note the following consequence of the Christoffel–Darboux formula:

$$p_n^2(x) = K_n^2(x, x_{kn})(x - x_{kn})^2 \Big/ \left[\frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x_{kn}) \right]^2.$$

Then the Cauchy–Schwarz inequality and (6.6) show that

$$p_n^2(x) \leqslant \lambda_n^{-1}(x)\, \lambda_n^{-1}(x_{kn})(x - x_{kn})^2 \Big/ \left[\frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x_{kn}) \right]^2$$

$$= \lambda_n^{-1}(x) \left[A_n(x_{kn}) \Big/ \frac{\gamma_{n-1}}{\gamma_n} \right] (x - x_{kn})^2. \tag{6.11}$$

Now if $x \geqslant 0$ and x_{kn} is the zero of $p_n(x)$ closest to x, then by Corollary 1.2 and (5.6), we have

$$(x - x_{kn})^2 \leqslant C_1 \Lambda_n^2(x_{kn}) \leqslant C_2 \Lambda_n^2(x).$$

Together with Theorem 1.1(a) and Corollary 1.2, this gives

$$p_n^2(x)\, W_\alpha^2(x) \leqslant C_3 \Lambda_n(x) \left[A_n(x_{kn}) \Big/ \frac{\gamma_{n-1}}{\gamma_n} \right] \tag{6.12}$$

$$\leqslant C_4 \Lambda_n(x)\, x_{kn}^{\alpha-2}, \tag{6.13}$$

by (6.8). Fix $\varepsilon \in (0, 1)$. We consider two ranges of x:

I. $x \in [\varepsilon a_n, \frac{1}{2} a_n]$. Here from (1.7),

$$A_n(x) \sim x^{1-\alpha},$$

so (6.13) becomes

$$(p_n W_\alpha)^2(x) \leqslant C_5/x \sim a_n^{-1},$$

as required.

II. $x \in [\frac{1}{2} a_n, a_n]$. Here from (1.8) and (6.13), we obtain

$$\begin{aligned}(p_n W_\alpha)^2(x) &\leqslant C_6 n^{1/\alpha - 1} \max\{n^{-2/3}, 1 - |x|/a_n\}^{-1/2} a_n^{\alpha - 2} \\ &\leqslant C_7 a_n^{-1} \max\{n^{-2/3}, 1 - |x|/a_n\}^{-1/2}. \quad \blacksquare\end{aligned}$$

Remark. Let $0 < \alpha < 1$. Note that if $x \in [0, \varepsilon a_n]$ and we choose x_{kn} to be the closest zero of p_n on the right of x, we have $x_{kn} \sim 1 + x$ because of the spacing (1.13) of the zeros. Then (6.13) becomes

$$(p_n W_\alpha)^2(x) \leqslant C_1 A_n(x)(1+x)^{\alpha-2} \leqslant C_2/(1+x), \qquad x \in [0, \varepsilon a_n]. \quad (6.14)$$

For $\alpha = 1$, (6.13) similarly becomes

$$(p_n W_\alpha)^2(x) \leqslant C_3 \left(\log \frac{\pi n}{1+x}\right)^{-1} \left(\frac{1}{\log n} + x\right)^{-1}, \qquad x \in [0, \varepsilon a_n]. \quad (6.15)$$

At least for $\alpha = 1$, we can improve (6.15) a little:

Proof of Corollary 1.5. Let $\beta \in (0, \frac{1}{2})$ and define

$$h_n(x) := a_n x^\beta (p_n W_1)^2(x).$$

From (6.7), we obtain for $x \leqslant a_n/2$ (recall that $\alpha = 1$),

$$\begin{aligned}A_n(x) \Big/ \frac{\gamma_{n-1}}{\gamma_n} &\leqslant C_1 \left[x^{-1} a_n^{-1} \int_0^x h_n(t)\, t^{-\beta}\, dt + a_n^{-1} \int_x^{a_n/2} h_n(t)\, t^{-1-\beta}\, dt \right. \\ &\qquad \left. + \int_{a_n/2}^{a_n} (p_n W_1)^2(t)\, t^{-1}\, dt\right] \\ &\leqslant C_2 [x^{-1} a_n^{-1} x^{1-\beta} \|h_n\|_{L_\infty[0, a_n/2]} + a_n^{-1} x^{-\beta} \|h_n\|_{L_\infty[0, a_n/2]} + a_n^{-1}] \\ &< C_3 a_n^{-1} (x^{-\beta} \|h_n\|_{L_\infty[0, a_n/2]} + 1).\end{aligned}$$

Then we obtain from (6.12), if we choose x_{kn} to the right of x, that for $x \leqslant a_n/2$,

$$h_n(x) \leqslant C_4 \left[\log \frac{\pi n}{1+x}\right]^{-1} (\|h_n\|_{L_\infty[0,\, a_n/2]} + x^\beta)$$
$$\leqslant \frac{1}{2} \|h_n\|_{L_\infty[0,\, a_n/2]} + a_n^\beta,$$

if $x \in [0, \varepsilon a_n]$, and ε is small enough. Then

$$\|h_n\|_{L_\infty[0,\, \varepsilon a_n]} \leqslant \tfrac{1}{2} \|h_n\|_{L_\infty[0,\, a_n/2]} + a_n^\beta.$$

Recall, from our bounds in Corollary 1.4, that

$$\|h_n\|_{L_\infty[\varepsilon a_n,\, a_n/2]} \leqslant C_5 a_n^\beta.$$

Then we deduce that

$$\|h_n\|_{L_\infty[0,\, \varepsilon a_n]} \leqslant \tfrac{1}{2} \|h_n\|_{L_\infty[0,\, \varepsilon a_n]} + C_6 a_n^\beta,$$

and hence that

$$(p_n W_1)^2 (x) \leqslant C_7 a_n^{-1} \left(\frac{a_n}{x}\right)^\beta, \qquad x \in [0, \varepsilon a_n].$$

Since $\beta > 0$ is arbitrary, we deduce that given $\delta > 0$,

$$(p_n W_1)^2 (x) < C_8 a_n^{-1} n^\delta, \qquad x \in [a_n^{-1}, \varepsilon a_n].$$

To fill in the interval $[0, a_n^{-1}]$, we use the bound

$$\| p_n W_1 \|_{L_\infty(\mathbb{R})}^2 \leqslant C_9 \log n,$$

which is an easy consequence of the Christoffel function estimates of Theorem 1.1. Moreover, we need the Markov inequality [20, 9],

$$\| p_n' W_1 \|_{L_\infty(\mathbb{R})} \leqslant C_{10} \log n \, \| p_n W_1 \|_{L_\infty(\mathbb{R})} \leqslant C_{11} (\log n)^{3/2}.$$

Then, given $x \in [0, a_n^{-1}]$, we deduce that for some $\xi \in [x, a_n^{-1}]$,

$$p_n(x) = p_n(a_n^{-1}) + p_n'(\xi)(x - a_n^{-1}) = O(a_n^{-1/2} n^\delta) + O(\log n)^{3/2} \cdot O(a_n^{-1})$$
$$= O(a_n^{-1/2} n^\delta),$$

if δ is small enough. We have shown that for any given $\delta > 0$,

$$\| p_n W_1 \|_{L_\infty[0, \varepsilon a_n]} \leqslant C_{12} a_n^{-1/2} n^{\delta}. \tag{6.16}$$

Also, our upper bound in Corollary 1.4 gives

$$| p_n W_1 |(x) \leqslant C_{13} a_n^{-1/2} n^{1/6}, \ |x| \in [\varepsilon a_n, a_n(1 - n^{-2/3})].$$

The infinite–finite range inequality Lemma 3.1 gives

$$\| p_n W_1 \|_{L_\infty(\mathbb{R})} \leqslant C_{14} a_n^{-1/2} n^{1/6}. \tag{6.17}$$

In the other direction, we use (6.5), (6.6), which give

$$\begin{aligned} \lambda_{jn}^{-1} W^2(x_{jn}) &= \left[A_n(x_{jn}) \Big/ \frac{\gamma_{n-1}}{\gamma_n} \right]^{-1} (p_n' W_1)^2 (x_{jn}) \\ &= \left[A_n(x_{jn}) \Big/ \frac{\gamma_{n-1}}{\gamma_n} \right]^{-1} ((p_n W_1)' (x_{jn}))^2. \end{aligned}$$

Then applying our estimates of Theorem 1.1 and Lemma 6.1 gives for $x_{jn} \geqslant \varepsilon a_n$,

$$((p_n W_1)' (x_{jn}))^2 \sim a_n^{-1} \max\{n^{-2/3}, 1 - |x_{jn}|/a_n\}^{1/2}. \tag{6.18}$$

(Recall that $a_n \sim n$.) Applying the Markov–Bernstein inequality Theorem 1.3 in [9, p. 1067] gives

$$|(p_n W_1)' (x_{jn})| \leqslant C_{15} \max\{n^{-2/3}, 1 - |x_{jn}|/a_n\}^{1/2} \| p_n W_1 \|_{L_\infty(\mathbb{R})}.$$

Combining this with (6.18), setting $j = 1$, and using Corollary 1.2(a) give

$$\| p_n W_1 \|_{L_\infty(\mathbb{R})} \geqslant C_{16} a_n^{-1/2} n^{1/6}.$$

Together with (6.17), this gives the result. ∎

Proof of Corollary 1.3. First, we remark that proceeding exactly as before (6.18) gives for $0 < \alpha \leqslant 1$, and $x_{jn} \geqslant \varepsilon a_n$,

$$\begin{aligned} \frac{a_n}{n} | p_n' W_\alpha(x_{jn})| &= \frac{a_n}{n} |(p_n W_\alpha)' (x_{jn})| \\ &\sim a_n^{-1/2} \max\{n^{-2/3}, 1 - |x_{jn}|/a_n\}^{+1/4}. \end{aligned} \tag{6.19}$$

Now it is known [12] that

$$\frac{\gamma_{n-1}}{\gamma_n} \sim a_n,$$

so from (6.8),

$$A_n(x_{jn}) \sim a_n^{\alpha-1} \sim n/a_n$$

for this range of j. Then (6.5) gives

$$|p_{n-1} W_\alpha|(x_{jn}) \sim \frac{a_n}{n} |p_n' W_\alpha|(x_{jn}),$$

and this completes the proof of the corollary. ▮

REFERENCES

1. F. F. ABI-KHUZAM, The asymptotic behaviour of a Lindelof function and its Taylor coefficients, *J. Math. Appl.* **93** (1983), 495–526.
2. N. I. AKHIEZER, "The Classical Moment Problem" (N. Kemmer, Transl.), Oliver & Boyd, Edinburgh, 1965.
3. C. BERG AND H. L. PEDERSEN, On the order and type of the entire functions associated with an indeterminate Hamburger moment problem, preprint.
4. G. CRISCUOLO, B. DELLA VECCHIA, D. S. LUBINSKY, AND G. MASTROIANNI, Functions of the second kind for Freud weights and series expansions of Hilbert transforms, *J. Math. Anal. Appl.*, to appear.
5. G. FREUD, "Orthogonal Polynomials," Akademiai Kaido/Pergamon, Budapest, 1971.
6. G. FREUD, On estimations of the greatest zeros of orthogonal polynomials, *Acta Math. Acad. Sci. Hungar.* **25** (1974), 99–107.
7. G. FREUD, A. GIROUX, AND Q. I. RAHMAN, On approximation by polynomials with weight $\exp(-|x|)$, *Canad. J. Math.* **30** (1978), 358–372. [in French].
8. A. KNOPFMACHER AND D. S. LUBINSKY, Mean convergence of Lagrange interpolation for Freud's weights with application to product integration rules, *J. Comput. Appl. Math.* **17** (1987), 79–103.
9. A. L. LEVIN AND D. S. LUBINSKY, L_∞ Markov and Bernstein inequalities for Freud weights, *SIAM J. Math. Anal.* **21** (1990), 1065–1082.
10. A. L. LEVIN AND D. S. LUBINSKY, Christoffel functions, orthogonal polynomials and Nevai's conjecture for Freud weights, *Constr. Approx.* **8** (1992), 463–535.
11. D. S. LUBINSKY, "Strong Asymptotics for Extremal Errors and Polynomials Associated with Erdös-Type Weights," Pitman Research Notes in Mathematics, Vol. 202, Longman, Harlow, Essex, 1989.
12. D. S. LUBINSKY AND E. B. SAFF, "Strong Asymptotics for Extremal Polynomials Associated with Exponential Weights," Lecture Notes in Mathematics, Vol. 1305, Springer-Verlag, Berlin/New York, 1988.
13. D. S. LUBINSKY AND V. TOTIK, How to discretise a logarithmic potential, *Acta Math. Sci. Hungar.*, to appear.

14. A. Maté, P. Nevai, and V. Totik, Asymptotics for the zeros of orthogonal polynomials associated with infinite intervals, *J. London Math. Soc.* **33** (1986), 303–310.
15. H. N. Mhaskar, Bounds for certain Freud-type orthogonal polynomials, *J. Approx. Theory* **63** (1990), 238–254.
16. H. N. Mhaskar and E. B. Saff, Extremal problems for polynomials with exponential weights, *Trans. Amer. Math. Soc.* **285** (1984), 203–234.
17. H. N. Mhaskar and E. B. Saff, Where does the sup-norm of a weighted polynomial live? *Constr. Approx.* **1** (1985), 71–91.
18. P. Nevai, "Orthogonal Polynomials," Memoirs Amer. Math. Soc., Vol. 213, Amer. Math. Soc., Providence, RI, 1979.
19. P. Nevai, Geza Freud, Orthogonal polynomials and Christoffel functions. A case study, *J. Approx. Theory* **48** (1986), 3–167.
20. P. Nevai and V. Totik, Weighted polynomial inequalities, *Constr. Approx.* **2** (1986), 113–127.
21. E. A. Rahmanov, On asymptotic properties of polynomials orthogonal on the real axis, *Math. USSR.-Sb.* **47** (1984), 155–193.
22. E. A. Rahmanov, Strong asymptotics for orthogonal polynomials, in preparation.
23. G. A. Szegö, "Orthogonal Polynomials," Colloquium Publications, Vol. 23, 4th ed. Amer. Math. Soc., Providence, RI, 1975, 1st ed., 1939.
24. V. Totik, "Weighted Approximation with Varying Weights," Springer Lecture Notes in Mathematics, Vol. 1569, Springer, Berlin, 1994.